13-3 Sum and Difference of Angles Identities

Find the exact value of each expression.

1. $\cos 165°$

SOLUTION:

$$\begin{aligned}\cos 165° &= \cos(120° + 45°)\\ &= \cos 120° \cos 45° - \sin 120° \sin 45°\\ &= \cos(90° + 30°)\cos 45° - \sin(90° + 30°)\sin 45°\\ &= -\sin 30° \cos 45° - \cos 30° \sin 45°\\ &= -\frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}\\ &= \frac{-(1+\sqrt{3})}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}\\ &= -\frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

2. $\cos 105°$

SOLUTION:

$$\begin{aligned}\cos 105° &= \cos(60° + 45°)\\ &= \cos 60° \cos 45° - \sin 60° \sin 45°\\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}\\ &= \frac{1-\sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}-\sqrt{6}}{4}\end{aligned}$$

3. $\cos 75°$

SOLUTION:

$$\begin{aligned}\cos 75° &= \cos(30° + 45°)\\ &= \cos 30° \cos 45° - \sin 30° \sin 45°\\ &= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

4. $\sin(-30°)$

SOLUTION:

$$\begin{aligned}\sin(-30°) &= -\sin 30° \\ &= -\frac{1}{2}\end{aligned}$$

5. $\sin 135°$

SOLUTION:

$$\begin{aligned}\sin 135° &= \sin(90° + 45°) \\ &= \sin 90° \cos 45° + \cos 90° \sin 45° \\ &= 1\left(\frac{1}{\sqrt{2}}\right) + 0\left(\frac{1}{\sqrt{2}}\right) \\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{\sqrt{2}}{2}\end{aligned}$$

6. $\sin(-210°)$

SOLUTION:

$$\begin{aligned}\sin(-210°) &= -\sin 210° \\ &= -\sin(180° + 30°) \\ &= -(\sin 180° \cos 30° + \cos 180° \sin 30°) \\ &= -\left(0\left(\frac{\sqrt{3}}{2}\right) + (-1)\frac{1}{2}\right) \\ &= \frac{1}{2}\end{aligned}$$

7. **CCSS MODELING** Refer to the beginning of the lesson. *Constructive interference* occurs when two waves combine to have a greater amplitude than either of the component waves. *Destructive interference* occurs when the component waves combine to have a smaller amplitude. The first signal can be modeled by the equation $y = 20\sin(3\theta + 45°)$. The second signal can be modeled by the equation $y = 20\sin(3\theta + 225°)$

a. Find the sum of the two functions.

b. What type of interference results when signals modeled by the two equations are combined?

SOLUTION:

a.

$$\begin{aligned} y + y &= 20\sin(3\theta + 225°) + 20\sin(3\theta + 45°) \\ &= 20\sin(180° + (3\theta + 45°)) + 20\sin(3\theta + 45°) \\ &= 20\sin 180° \cos(3\theta + 45°) + 20\cos 180° \sin(3\theta + 45°) + 20\sin(3\theta + 45°) \\ &= 20(0)\cos(3\theta + 45°) + 20(-1)\sin(3\theta + 45°) + 20\sin(3\theta + 45°) \\ &= -20\sin(3\theta + 45°) + 20\sin(3\theta + 45°) \\ &= 0 \end{aligned}$$

The sum of the two functions is 0.

b. The interference is destructive. The signals cancel each other completely.

Verify that each equation is an identity.

8. $\sin(90° + \theta) = \cos\theta$

SOLUTION:

$$\begin{aligned} \sin(90° + \theta) &\stackrel{?}{=} \cos\theta \\ \sin 90° \cos\theta + \cos 90° \sin\theta &\stackrel{?}{=} \cos\theta \\ 1 \cdot \cos\theta + 0 \cdot \sin\theta &\stackrel{?}{=} \cos\theta \\ \cos\theta &= \cos\theta \checkmark \end{aligned}$$

9. $\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

SOLUTION:

$$\begin{aligned} \cos\left(\frac{3\pi}{2} - \theta\right) &\stackrel{?}{=} -\sin\theta \\ \cos\frac{3\pi}{2}\cos\theta + \sin\frac{3\pi}{2}\sin\theta &\stackrel{?}{=} -\sin\theta \\ 0 \cdot \cos\theta - 1 \cdot \sin\theta &\stackrel{?}{=} -\sin\theta \\ -\sin\theta &= -\sin\theta \checkmark \end{aligned}$$

13-3 Sum and Difference of Angles Identities

10. $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot\theta$

SOLUTION:

$$\tan\left(\theta + \frac{\pi}{2}\right) = -\cot\theta$$

$$\frac{\sin\left(\theta + \frac{\pi}{2}\right)}{\cos\left(\theta + \frac{\pi}{2}\right)} \stackrel{?}{=} -\cot\theta$$

$$\frac{\sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2}}{\cos\theta\cos\frac{\pi}{2} - \sin\theta\sin\frac{\pi}{2}} \stackrel{?}{=} -\cot\theta$$

$$\frac{(\sin\theta)\cdot 0 + (\cos\theta)\cdot 1}{(\cos\theta)\cdot 0 - (\sin\theta)\cdot 1} \stackrel{?}{=} -\cot\theta$$

$$-\frac{\cos\theta}{\sin\theta} \stackrel{?}{=} -\cot\theta$$

$$-\cot\theta = -\cot\theta \checkmark$$

11. $\sin(\theta + \pi) = -\sin\theta$

SOLUTION:

$$\sin(\theta + \pi) \stackrel{?}{=} -\sin\theta$$

$$\sin\theta\cos\pi + \cos\theta\sin\pi \stackrel{?}{=} -\sin\theta$$

$$(\sin\theta)(-1) + (\cos\theta)(0) \stackrel{?}{=} -\sin\theta$$

$$-\sin\theta = -\sin\theta \checkmark$$

Find the exact value of each expression.

12. $\sin 165°$

SOLUTION:

$$\begin{aligned}\sin 165° &= \sin(120° + 45°)\\ &= \sin 120°\cos 45° + \cos 120°\sin 45°\\ &= \sin(90° + 30°)\cos 45° + \cos(90° + 30°)\sin 45°\\ &= \cos 30°\cos 45° - \sin 30°\sin 45°\\ &= \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}} - \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

13-3 Sum and Difference of Angles Identities

13. $\cos 135°$

SOLUTION:

$$\begin{aligned}\cos 135° &= \cos(90° + 45°)\\ &= \cos 90° \cos 45° - \sin 90° \sin 45°\\ &= 0 \cdot \frac{1}{\sqrt{2}} - 1 \cdot \frac{1}{\sqrt{2}}\\ &= -\frac{1}{\sqrt{2}} \text{ or } -\frac{\sqrt{2}}{2}\end{aligned}$$

14. $\cos \frac{7\pi}{12}$

SOLUTION:

$$\begin{aligned}\cos \frac{7\pi}{12} &= \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right)\\ &= \cos \frac{\pi}{3} \cos \frac{\pi}{4} - \sin \frac{\pi}{3} \sin \frac{\pi}{4}\\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}\\ &= \frac{1 - \sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\end{aligned}$$

15. $\sin \frac{\pi}{12}$

SOLUTION:

$$\begin{aligned}\sin \frac{\pi}{12} &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right)\\ &= \sin \frac{\pi}{3} \cos \frac{\pi}{4} - \cos \frac{\pi}{3} \sin \frac{\pi}{4}\\ &= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{3} - 1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

13-3 Sum and Difference of Angles Identities

16. $\tan 195°$

SOLUTION:

$$\begin{aligned}
\tan 195° &= \tan(135° + 60°) \\
&= \frac{\tan 135° + \tan 60°}{1 - \tan 135° \tan 60°} \\
&= \frac{\tan(90° + 45°) + \tan 60°}{1 - \tan(90° + 45°)\tan 60°} \\
&= \frac{-\cot 45° + \tan 60°}{1 + \cot 45° \tan 60°} \\
&= \frac{-1 + \sqrt{3}}{1 + 1\left(\sqrt{3}\right)} \\
&= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} - 1}{\sqrt{3} - 1} \\
&= \frac{3 - 2\sqrt{3} + 1}{\left(\sqrt{3}\right)^2 - 1^2} \\
&= \frac{4 - 2\sqrt{3}}{2} \\
&= 2 - \sqrt{3}
\end{aligned}$$

17. $\cos\left(-\frac{\pi}{12}\right)$

SOLUTION:

$$\begin{aligned}
\cos\left(-\frac{\pi}{12}\right) &= \cos\left(\frac{\pi}{12}\right) \\
&= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\
&= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4} \\
&= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} \\
&= \frac{1 + \sqrt{3}}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{2} + \sqrt{6}}{4}
\end{aligned}$$

18. **ELECTRONICS** In a certain circuit carrying alternating current, the formula $c = 2\sin(120t)$ can be used to find the current c in amperes after t seconds.
a. Rewrite the formula using the sum of two angles.
b. Use the sum of angles formula to find the exact current at $t = 1$ second.

SOLUTION:

a.

$$\begin{aligned} c &= 2\sin(120t) \\ &= 2\sin(90t + 30t) \end{aligned}$$

b. Substitute 1 for t and evaluate.

$$\begin{aligned} c &= 2\sin 120(1) \\ &= 2\sin 120 \\ &= 2\sin(90 + 30) \\ &= 2(\sin 90 \cos 30 + \cos 30 \sin 90) \\ &= 2\left(1 \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot 0\right) \\ &= \sqrt{3} \end{aligned}$$

The exact current at $t = 1$ second is $\sqrt{3}$ amperes.

Verify that each equation is an identity.

19. $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

SOLUTION:

$$\begin{aligned} \cos\left(\frac{\pi}{2} + \theta\right) &\stackrel{?}{=} -\sin\theta \\ \cos\frac{\pi}{2}\cos\theta - \sin\frac{\pi}{2}\sin\theta &\stackrel{?}{=} -\sin\theta \\ 0 \cdot \cos\theta - 1 \cdot \sin\theta &\stackrel{?}{=} -\sin\theta \\ -\sin\theta &= -\sin\theta \checkmark \end{aligned}$$

20. $\cos(60° + \theta) = \sin(30° - \theta)$

SOLUTION:

$$\begin{aligned} \cos(60° + \theta) &\stackrel{?}{=} \sin(30° - \theta) \\ \cos 60° \cos\theta - \sin 60° \sin\theta &\stackrel{?}{=} \sin 30° \cos\theta - \cos 30° \sin\theta \\ \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta &= \frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta \checkmark \end{aligned}$$

21. $\cos(180° + \theta) = -\cos\theta$

SOLUTION:

$$\cos(180° + \theta) \stackrel{?}{=} -\cos\theta$$
$$\cos 180° \cos\theta - \sin 180° \sin\theta \stackrel{?}{=} -\cos\theta$$
$$-1 \cdot \cos\theta - 0 \cdot \sin\theta \stackrel{?}{=} -\cos\theta$$
$$-\cos\theta = -\cos\theta \checkmark$$

22. $\tan(\theta + 45°) = \dfrac{1 + \tan\theta}{1 - \tan\theta}$

SOLUTION:

$$\tan(\theta + 45°) \stackrel{?}{=} \frac{1 + \tan\theta}{1 - \tan\theta}$$
$$\frac{\tan\theta + \tan 45°}{1 - \tan\theta \tan 45°} \stackrel{?}{=} \frac{1 + \tan\theta}{1 - \tan\theta}$$
$$\frac{\tan\theta + 1}{1 - (\tan\theta)(1)} \stackrel{?}{=} \frac{1 + \tan\theta}{1 - \tan\theta}$$
$$\frac{1 + \tan\theta}{1 - \tan\theta} = \frac{1 + \tan\theta}{1 - \tan\theta} \checkmark$$

23. **CCSS REASONING** The monthly high temperatures for Minneapolis, Minnesota, can be modeled by the equation $y = 31.65\sin\left(\frac{\pi}{6}x - 2.09\right) + 52.35$, where the months x are represented by January = 1, February = 2, and so on. The monthly low temperatures for Minneapolis can be modeled by the equation $y = 30.15\sin\left(\frac{\pi}{6}x - 2.09\right) + 32.95.$

a. Write a new function by adding the expressions on the right side of each equation and dividing the result by 2.
b. What is the meaning of the function you wrote in part a?

SOLUTION:

a.

$$2y = 31.65\sin\left(\frac{\pi}{6}x - 2.09\right) + 52.35 + 30.15\sin\left(\frac{\pi}{6}x - 2.09\right) + 32.95$$
$$= 61.8\sin\left(\frac{\pi}{6}x - 2.09\right) + 85.3$$
$$y = 30.9\sin\left(\frac{\pi}{6}x - 2.09\right) + 42.65$$

b. The new function represents the average of the high and low temperatures for each month.

Find the exact value of each expression.

24. $\tan 165°$

SOLUTION:

$$
\begin{aligned}
\tan 165° &= \tan(120° + 45°) \\
&= \frac{\tan 120° + \tan 45°}{1 - \tan 120° \tan 45°} \\
&= \frac{\tan(90° + 30°) + \tan 45°}{1 - \tan(90° + 30°)\tan 45°} \\
&= \frac{-\cot 30° + \tan 45°}{1 + \cot 30° \tan 45°} \\
&= \frac{-\sqrt{3} + 1}{1 + \sqrt{3}(1)} \\
&= \frac{1 - \sqrt{3}}{1 + \sqrt{3}} \\
&= \frac{1 - \sqrt{3}}{1 + \sqrt{3}} \cdot \frac{1 - \sqrt{3}}{1 - \sqrt{3}} \\
&= \sqrt{3} - 2
\end{aligned}
$$

13-3 Sum and Difference of Angles Identities

25. $\sec 1275°$

SOLUTION:

$$\begin{aligned}
\sec 1275° &= \sec(1440° - 165°) \\
&= \frac{1}{\cos(1440° - 165°)} \\
&= \frac{1}{\cos 1440° \cos 165° + \sin 1440° \sin 165°} \\
&= \frac{1}{1 \cdot \cos 165° + 0 \cdot \sin 165°} \\
&= \frac{1}{\cos 165°} \\
&= \frac{1}{\cos(120° + 45°)} \\
&= \frac{1}{\cos(90° + 30°)\cos 45° - \sin(90° + 30°)\sin 45°} \\
&= \frac{1}{-\sin 30° \cos 45° - \cos 30° \sin 45°} \\
&= \frac{1}{-\left(\frac{1}{2}\right) \cdot \left(\frac{1}{\sqrt{2}}\right) - \left(\frac{\sqrt{3}}{2}\right) \cdot \left(\frac{1}{\sqrt{2}}\right)} \\
&= \frac{1}{-\frac{\sqrt{2} + \sqrt{6}}{4}} \\
&= -\frac{4}{\sqrt{2} + \sqrt{6}} \\
&= -\frac{4}{\sqrt{2} + \sqrt{6}} \cdot \frac{\sqrt{2} - \sqrt{6}}{\sqrt{2} - \sqrt{6}} \\
&= \sqrt{2} - \sqrt{6}
\end{aligned}$$

13-3 Sum and Difference of Angles Identities

26. $\sin 735°$

SOLUTION:

$$\begin{aligned}
\sin 735° &= \sin(720° + 15°) \\
&= \sin 720° \cos 15° + \sin 15° \cos 720° \\
&= (0)\cos 15° + \sin 15°(1) \\
&= \sin 15° \\
&= \sin(45° - 30°) \\
&= \sin 45° \cos 30° - \sin 30° \cos 45° \\
&= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \\
&= \frac{\sqrt{3} - 1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{\sqrt{6} - \sqrt{2}}{4}
\end{aligned}$$

13-3 Sum and Difference of Angles Identities

27. $\tan \frac{23\pi}{12}$

SOLUTION:

$$\begin{aligned}
\tan \frac{23\pi}{12} &= \tan\left(2\pi - \frac{\pi}{12}\right) \\
&= \frac{\tan 2\pi - \tan \frac{\pi}{12}}{1 + \tan 2\pi \tan \frac{\pi}{12}} \\
&= \frac{0 - \tan \frac{\pi}{12}}{1 - 0 \cdot \tan \frac{\pi}{12}} \\
&= -\tan \frac{\pi}{12} \\
&= -\tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\
&= -\frac{\tan \frac{\pi}{4} - \tan \frac{\pi}{6}}{1 + \tan \frac{\pi}{4} \tan \frac{\pi}{6}} \\
&= -\frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} \\
&= -\frac{\sqrt{3} - 1}{\sqrt{3} + 1} \\
&= -\frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} - 1}{\sqrt{3} - 1} \\
&= \sqrt{3} - 2
\end{aligned}$$

28. $\csc\frac{5\pi}{12}$

SOLUTION:

$$\begin{aligned}\csc\frac{5\pi}{12} &= \frac{1}{\sin\left(\frac{\pi}{4}+\frac{\pi}{6}\right)}\\ &= \frac{1}{\sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6}}\\ &= \frac{1}{\frac{1}{\sqrt{2}}\cdot\frac{1}{2}+\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}}\\ &= \frac{4}{\sqrt{2}+\sqrt{6}}\\ &= \frac{4}{\sqrt{2}+\sqrt{6}}\cdot\frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}-\sqrt{6}}\\ &= \sqrt{6}-\sqrt{2}\end{aligned}$$

29. $\cot\frac{113\pi}{12}$

SOLUTION:

$$\begin{aligned}\cot\frac{113\pi}{12} &= \cot\left(10\pi-\frac{7\pi}{12}\right)\\ &= -\cot\frac{7\pi}{12}\\ &= -\frac{1}{\tan\left(\frac{\pi}{4}+\frac{\pi}{3}\right)}\\ &= -\frac{1}{\frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{3}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{3}}}\\ &= -\frac{1-1\cdot\sqrt{3}}{1+\sqrt{3}}\\ &= -\frac{1-\sqrt{3}}{1+\sqrt{3}}\cdot\frac{1-\sqrt{3}}{1-\sqrt{3}}\\ &= 2-\sqrt{3}\end{aligned}$$

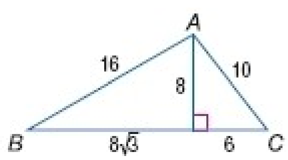
30. **FORCE** In the figure given below, the effort F necessary to hold a safe in position non a ramp is given by $F = \dfrac{W(\sin A + \mu \cos A)}{\cos A - \mu \sin A}$, where W is the weight of the safe and $\mu = \tan\theta$. Show that $F = W\tan(A+\theta)$.

SOLUTION:

$$\begin{aligned}
F &= \frac{W(\sin A + \mu \cos A)}{\cos A - \mu \sin A} \\
&= \frac{W(\sin A + \tan\theta \cos A)}{\cos A - \tan\theta \sin A} \\
&= \frac{W\left(\dfrac{\sin A}{\cos A} + \dfrac{\tan\theta \cos A}{\cos A}\right)}{\dfrac{\cos A}{\cos A} - \dfrac{\tan\theta \sin A}{\cos A}} \\
&= \frac{W(\tan A + \tan\theta)}{1 - \tan A \tan\theta} \\
&= W\tan(A+\theta)
\end{aligned}$$

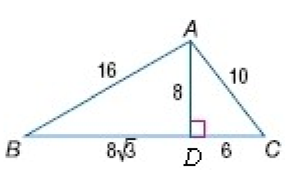
31. **QUILTING** As part of a quilt that is being made, the quilter places two right triangular swatches together to make a new triangular piece. One swatch has sides 6 inches, 8 inches, and 10 inches long. The other swatch has sides 8 inches, $8\sqrt{3}$ inches, and 16 inches long. The pieces are placed with the sides of eight inches against each other, as shown I the figure, to form triangle ABC.

a. What is the exact value of the sine of angle BAC?
b. What is the exact value of the cosine of angle BAC?
c. What is the measure of angle BAC?
d. Is the new triangle formed from the two triangles also a right triangle?



SOLUTION:

a. Name the vertices.



Consider the triangle ADC.

$$\sin \angle DAC = \frac{6}{10}$$
$$\cos \angle DAC = \frac{8}{10}$$

Consider the triangle ABD.

$$\sin \angle BAD = \frac{8\sqrt{3}}{16}$$
$$\cos \angle BAD = \frac{8}{16}$$

$$\begin{aligned}\sin \angle BAC &= \sin(\angle DAC + \angle BAD)\\ &= \sin \angle DAC \cos \angle BAD + \cos \angle DAC \sin \angle BAD\\ &= \frac{6}{10} \cdot \frac{8}{16} + \frac{8}{10} \cdot \frac{8\sqrt{3}}{16}\\ &= \frac{48 + 64\sqrt{3}}{160}\\ &= \frac{3 + 4\sqrt{3}}{10}\end{aligned}$$

b.

$$\begin{aligned}\cos \angle BAC &= \cos(\angle DAC + \angle BAD)\\ &= \cos \angle DAC \cos \angle BAD - \sin \angle DAC \sin \angle BAD\\ &= \frac{8}{10} \cdot \frac{8}{16} - \frac{6}{10} \cdot \frac{8\sqrt{3}}{16}\\ &= \frac{64 - 48\sqrt{3}}{160}\\ &= \frac{4 - 3\sqrt{3}}{10}\end{aligned}$$

c.

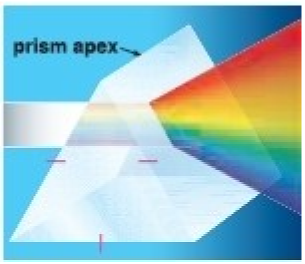
$$\cos \angle BAC = \frac{4 - 3\sqrt{3}}{10}$$
$$\begin{aligned}\angle BAC &= \cos^{-1}\left(\frac{4 - 3\sqrt{3}}{10}\right)\\ &= 96.9°\end{aligned}$$

d. Since the angle *BAC* is greater than 90°, the triangle *BAC* is not a right triangle.

32. **OPTICS** When light passes symmetrically through a prism, the index of refraction n of the glass with respect to air is $n = \dfrac{\sin\left[\frac{1}{2}(a+b)\right]}{\sin\frac{b}{2}}$, where a is the measure of the deviation angle and b is the measure of the prism apex angle.

a. Show that for the prism shown, $n = \sqrt{3}\sin\frac{a}{2} + \cos\frac{a}{2}$

b. Find n for the prism shown.



SOLUTION:

a.

Since the bases of the prism is the equilateral triangle, the measure of the prism apex angle is $60°$.

$$\begin{aligned}
\frac{\sin\left[\frac{1}{2}(a+b)\right]}{\sin\frac{b}{2}} &= \frac{\sin\left[\frac{1}{2}(a+60^\circ)\right]}{\sin\frac{60^\circ}{2}} \\
&= \frac{\sin\left(\frac{a}{2}+30^\circ\right)}{\sin 30^\circ} \\
&= \frac{\sin\frac{a}{2}\cos 30^\circ + \cos\frac{a}{2}\sin 30^\circ}{\sin 30^\circ} \\
&= \frac{\left(\sin\frac{a}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\cos\frac{a}{2}\right)\left(\frac{1}{2}\right)}{\frac{1}{2}} \\
&= \sqrt{3}\sin\frac{a}{2} + \cos\frac{a}{2}
\end{aligned}$$

b.

Here $b = 180°$

$$\begin{aligned}
n &= \sqrt{3}\sin\frac{a}{2} + \cos\frac{a}{2} \\
&= \sqrt{3}\sin\frac{180°}{2} + \cos\frac{180°}{2} \\
&= \sqrt{3}
\end{aligned}$$

33. **MULTIPLE REPRESENTATIONS** In this problem, you will disprove the hypothesis that

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$\sin(A+B) = \sin A + \sin B$.

a. TABULAR Complete the table

A	B	$\sin A$	$\sin B$	$\sin(A+B)$	$\sin A + \sin B$
30°	90°				
45°	60°				
60°	45°				
90°	30°				

b. GRAPHICAL Assume that B is always 15° less than A. Use a graphing calculator to graph $\sin(x + x - 15)$ and $\sin x + \sin(x - 15)$ on the same screen.

c. ANALYTICAL Determine whether $\cos(A+B) = \cos A + \cos B$ is an identity. Explain your reasoning.

SOLUTION:

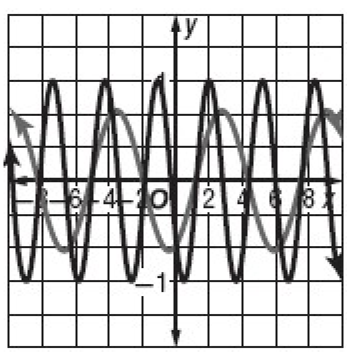
a.

A	B	$\sin A$	$\sin B$	$\sin(A+B)$	$\sin A + \sin B$
30°	90°	$\frac{1}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{3}{2}$
45°	60°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}+\sqrt{6}}{4}$	$\frac{\sqrt{2}+\sqrt{3}}{2}$
60°	45°	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}+\sqrt{6}}{4}$	$\frac{\sqrt{2}+\sqrt{3}}{2}$
90°	30°	1	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{3}{2}$

b.

KEYSTROKE:

Y= SIN ALPHA [X] + ALPHA [X] – 15) ENTER SIN ALPHA [X]) + SIN ALPHA [X] – 1 5 GRAPH .



c. No; a counterexample is $\cos(30° + 45°) = \cos 30° + \cos 45°$, which equals $\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}$ or about 1.5731. Since a

cosine value cannot be greater than 1, this statement must be false.

Verify that each equation is an identity.

34. $\sin(A+B)=\dfrac{\tan A+\tan B}{\sec A\sec B}$

SOLUTION:

$$\sin(A+B)\stackrel{?}{=}\frac{\tan A+\tan B}{\sec A\sec B}$$

$$\sin(A+B)\stackrel{?}{=}\frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{\frac{1}{\cos A}\cdot\frac{1}{\cos B}}$$

$$\sin(A+B)\stackrel{?}{=}\frac{\frac{\sin A}{\cos A}+\frac{\sin B}{\cos B}}{\frac{1}{\cos A}\cdot\frac{1}{\cos B}}\cdot\frac{\cos A\cos B}{\cos A\cos B}$$

$$\sin(A+B)\stackrel{?}{=}\frac{\sin A\cos B+\cos A\sin B}{1}$$

$$\sin(A+B)=\sin(A+B)\checkmark$$

35. $\cos(A+B)=\dfrac{1-\tan A\tan B}{\sec A\sec B}$

SOLUTION:

$$\cos(A+B)\stackrel{?}{=}\frac{1-\tan A\tan B}{\sec A\sec B}$$

$$\cos(A+B)\stackrel{?}{=}\frac{1-\frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}}{\frac{1}{\cos A}\cdot\frac{1}{\cos B}}$$

$$\cos(A+B)\stackrel{?}{=}\frac{1-\frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}}{\frac{1}{\cos A}\cdot\frac{1}{\cos B}}\cdot\frac{\cos A\cos B}{\cos A\cos B}$$

$$\cos(A+B)\stackrel{?}{=}\frac{\cos A\cos B-\sin A\sin B}{1}$$

$$\cos(A+B)=\cos(A+B)\checkmark$$

13-3 Sum and Difference of Angles Identities

36. $\sec(A-B)=\dfrac{\sec A\sec B}{1+\tan A\tan B}$

SOLUTION:

$$\sec(A-B)\stackrel{?}{=}\frac{\sec A\sec B}{1+\tan A\tan B}$$

$$\sec(A-B)\stackrel{?}{=}\frac{\frac{1}{\cos A}\cdot\frac{1}{\cos B}}{1+\frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}}$$

$$\sec(A-B)\stackrel{?}{=}\frac{\frac{1}{\cos A}\cdot\frac{1}{\cos B}}{1+\frac{\sin A}{\cos A}\cdot\frac{\sin B}{\cos B}}\cdot\frac{\cos A\cos B}{\cos A\cos B}$$

$$\sec(A-B)\stackrel{?}{=}\frac{1}{\cos A\cos B+\sin A\sin B}$$

$$\sec(A-B)\stackrel{?}{=}\frac{1}{\cos(A-B)}$$

$$\sec(A-B)=\sec(A-B)\checkmark$$

37. $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$

SOLUTION:

$$\sin(A+B)\sin(A-B)\stackrel{?}{=}\sin^2 A-\sin^2 B$$

$$(\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)\stackrel{?}{=}\sin^2 A-\sin^2 B$$

$$(\sin A\cos B)^2-(\cos A\sin B)^2\stackrel{?}{=}\sin^2 A-\sin^2 B$$

$$\sin^2 A\cos^2 B-\cos^2 A\sin^2 B\stackrel{?}{=}\sin^2 A-\sin^2 B$$

$$\sin^2 A\cos^2 B+\sin^2 A\sin^2 B-\sin^2 A\sin^2 B-\cos^2 A\sin^2 B\stackrel{?}{=}\sin^2 A-\sin^2 B$$

$$\sin^2 A\left(\cos^2 B+\sin^2 B\right)-\sin^2 B\left(\sin^2 A+\cos^2 A\right)\stackrel{?}{=}\sin^2 A-\sin^2 B$$

$$\left(\sin^2 A\right)(1)-\left(\sin^2 B\right)(1)\stackrel{?}{=}\sin^2 A-\sin^2 B$$

$$\sin^2 A-\sin^2 B=\sin^2 A-\sin^2 B\checkmark$$

38. **REASONING** Simplify the following expression without expanding any of the sums or differences.

$$\sin\left(\frac{\pi}{3}-\theta\right)\cos\left(\frac{\pi}{3}+\theta\right)-\cos\left(\frac{\pi}{3}-\theta\right)\sin\left(\frac{\pi}{3}+\theta\right)$$

SOLUTION:

$$\begin{aligned}\sin\left(\frac{\pi}{3}-\theta\right)\cos\left(\frac{\pi}{3}+\theta\right)-\cos\left(\frac{\pi}{3}-\theta\right)\sin\left(\frac{\pi}{3}+\theta\right)&=\sin\left(\left(\frac{\pi}{3}-\theta\right)-\left(\frac{\pi}{3}+\theta\right)\right)\\&=\sin\left(\frac{\pi}{3}-\theta-\frac{\pi}{3}-\theta\right)\\&=\sin(-2\theta)\end{aligned}$$

39. **WRITING IN MATH** Use the information at the beginning of the lesson and in Exercise 7 to explain how the sum and difference identities are used to describe wireless Internet interference. Include an explanation of the difference between constructive and destructive interference.

SOLUTION:

Sample answer: To determine wireless Internet Interference, you need to determine the sine or cosine of the sum or difference of two angles. Interference occurs when waves pass through the same space at the same time. When the combined waves have a greater amplitude, constructive interference results. When the combined waves have a smaller amplitude, destructive interference results.

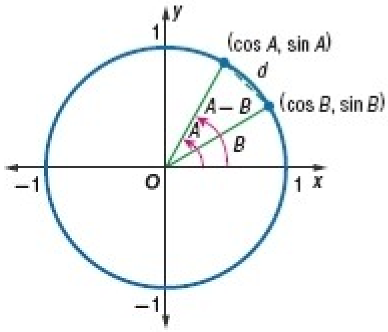
40. **CHALLENGE** Derive an identity for cot (A + B) in terms of cot A and cot B.

SOLUTION:

$$\begin{aligned}\cot(A+B)&=\frac{1}{\tan(A+B)}\\&=\frac{1}{\frac{\tan A+\tan B}{1-\tan A\tan B}}\\&=\frac{1-\tan A\tan B}{\tan A+\tan B}\\&=\frac{1-\frac{1}{\cot A}\cdot\frac{1}{\cot B}}{\frac{1}{\cot A}+\frac{1}{\cot B}}\cdot\frac{\cot A\cot B}{\cot A\cot B}\\&=\frac{\cot A\cot B-1}{\cot A+\cot B}\end{aligned}$$

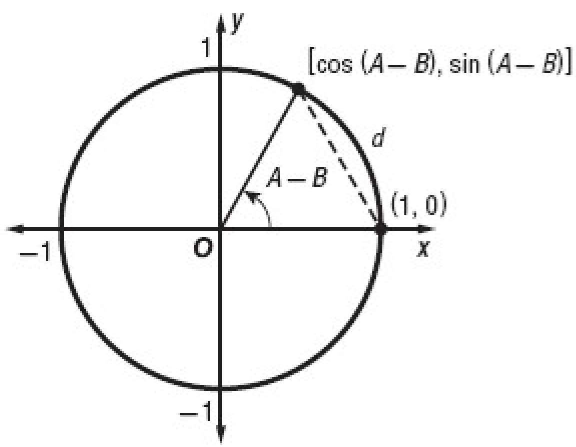
41. **CCSS ARGUMENTS** The figure shown has two angles A and B in standard position on the unit circle. Use the Distance Formula to find d, where (x_1, y_1) = (cos B, sin B) and (x_2, y_2) = (cos A, sin A).

13-3 Sum and Difference of Angles Identities



SOLUTION:

$$d=\sqrt{(\cos A-\cos B)^2+(\sin A-\sin B)^2}$$
$$d^2=(\cos A-\cos B)^2+(\sin A-\sin B)^2$$
$$d^2=\left(\cos^2 A-2\cos A\cos B+\cos^2 B\right)+\left(\sin^2 A-2\sin A\sin B+\sin^2 B\right)$$
$$d^2=\cos^2 A+\sin^2 A+\cos^2 B+\sin^2 B-2\cos A\cos B-2\sin A\sin B$$
$$d^2=1+1-2\cos A\cos B-2\sin A\sin B\left(\begin{array}{l}\text{since, }\sin^2 A+\cos^2 A=1\\ \text{and }\sin^2 B+\cos^2 B=1\end{array}\right)$$
$$d^2=2-2\cos A\cos B-2\sin A\sin B$$



Now find the value of d^2 when the angle having measure $A-B$ is in standard position on the unit circle, as shown in the figure at the left.

$$d=\sqrt{\left[\cos(A-B)-1\right]^2+\left[\sin(A-B)-0\right]^2}$$
$$d^2=\left[\cos(A-B)-1\right]^2+\left[\sin(A-B)-0\right]^2$$
$$d^2=\left[\cos^2(A-B)-2\cos(A-B)+1\right]+\sin^2(A-B)$$
$$d^2=\cos^2(A-B)+\sin^2(A-B)-2\cos(A-B)+1$$
$$d^2=1-2\cos(A-B)+1$$
$$d^2=2-2\cos(A-B)$$

13-3 Sum and Difference of Angles Identities

42. **OPEN ENDED** Consider the following theorem. *If A, B, and C are the angles of an oblique triangle, then tan A + tan B + tan C = tan A tan B tan C.* Choose values for *A, B,* and *C*. Verify that the conclusion is true for your specific values.

SOLUTION:

Sample answer :

A = 35°, B = 60°, C = 85°;

$$0.7002 + 1.7321 + 11.4301 \stackrel{?}{=} (0.7002) \cdot (1.7321)(11.4301)$$
$$13.86 = 13.86 \checkmark$$

43. **GRIDDED RESPONSE** The mean of seven numbers is 0. The sum of three of the numbers is –9. What is the sum of the remaining four numbers?

SOLUTION:

The sum of all the seven number is $9 \times 0 = 0$.
The sum of three numbers is –9.
Therefore, the sum of remaining four numbers is 0 – (–9) = 9.

44. The variables *a*, *b*, *c*, *d*, and *f* are integers in a sequence, where *a* = 2 and *b* = 12. To find the next term, double the last term and add that result to one less than the next-to-last term. For example, *c* = 25, because 2(12) = 24, 2 – 1 = 1, and 24 + 1 = 25. What is the value of *f* ?

A 74
B 144
C 146
D 256

SOLUTION:

Given *a* = 2, *b* = 12 and *c* = 25.

d = 2(25) + (12 – 1) = 61
f = 2(61) + (25 – 1) = 146

Option C is the correct answer.

45. **SAT/ACT** Solve $x^2 - 5x < 14$.
F $\{x | -7 < x < 2\}$
G $\{x | x < -7 \text{ or } x > 2\}$
H $\{x | -2 < x < 7\}$
J $\{x | x < -2 \text{ or } x > 7\}$
K $\{x | x > -2 \text{ and } x < 7\}$

SOLUTION:

$$\begin{aligned} x^2 - 5x &< 14 \\ x^2 - 5x - 14 &< 0 \\ (x-7)(x+2) &< 0 \end{aligned}$$

Therefore, the solution region is $\{x | -2 < x < 7\}$.
Option H is the correct answer.

46. **PROBABILITY** A math teacher is randomly distributing 15 yellow pencils and 10 green pencils. What is the probability that the first pencil she hands out will be yellow and the second pencil will be green?
A $\frac{1}{24}$
B $\frac{1}{4}$
C $\frac{2}{5}$
D $\frac{23}{25}$

SOLUTION:
The probability that the first pencil she hands out will be yellow and the second pencil will be green is $\frac{{}_{15}P_1 \cdot {}_{10}P_1}{{}_{25}P_2}$ or $\frac{1}{4}$.

Option B is the correct answer.

Verify that each equation is an identity.

47. $\frac{\sin\theta}{\tan\theta} + \frac{\cos\theta}{\cot\theta} = \cos\theta + \sin\theta$

SOLUTION:

$$\begin{aligned} \frac{\sin\theta}{\tan\theta} + \frac{\cos\theta}{\cot\theta} &\stackrel{?}{=} \cos\theta + \sin\theta \\ \frac{\sin\theta}{\frac{\sin\theta}{\cos\theta}} + \frac{\cos\theta}{\frac{\cos\theta}{\sin\theta}} &\stackrel{?}{=} \cos\theta + \sin\theta \\ \sin\theta \cdot \frac{\cos\theta}{\sin\theta} + \cos\theta \cdot \frac{\sin\theta}{\cos\theta} &\stackrel{?}{=} \cos\theta + \sin\theta \\ \cos\theta + \sin\theta &= \cos\theta + \sin\theta \checkmark \end{aligned}$$

48. $\sec\theta(\sec\theta - \cos\theta) = \tan^2\theta$

SOLUTION:

$$\begin{aligned}
\sec\theta(\sec\theta - \cos\theta) &\stackrel{?}{=} \tan^2\theta \\
\frac{1}{\cos\theta}\left(\frac{1}{\cos\theta} - \cos\theta\right) &\stackrel{?}{=} \tan^2\theta \\
\frac{1}{\cos^2\theta} - 1 &\stackrel{?}{=} \tan^2\theta \\
\sec^2\theta - 1 &\stackrel{?}{=} \tan^2\theta \\
\tan^2\theta &= \tan^2\theta \checkmark
\end{aligned}$$

Simplify each expression.

49. $\sin\theta\csc\theta - \cos^2\theta$

SOLUTION:

$$\begin{aligned}
\sin\theta\csc\theta - \cos^2\theta &= \sin\theta \cdot \frac{1}{\sin\theta} - \cos^2\theta \\
&= 1 - \cos^2\theta \\
&= \sin^2\theta
\end{aligned}$$

50. $\cos^2\theta\sec\theta\csc\theta$

SOLUTION:

$$\begin{aligned}
\cos^2\theta\sec\theta\csc\theta &= \frac{\cos^2\theta}{\cos\theta\sin\theta} \\
&= \frac{\cos\theta}{\sin\theta} \\
&= \cot\theta
\end{aligned}$$

51. $\cos\theta + \sin\theta\tan\theta$

SOLUTION:

$$\begin{aligned}
\cos\theta + \sin\theta\tan\theta &= \cos\theta + \sin\theta\frac{\sin\theta}{\cos\theta} \\
&= \frac{\cos^2\theta + \sin^2\theta}{\cos\theta} \\
&= \frac{1}{\cos\theta} \\
&= \sec\theta
\end{aligned}$$

52. **GUITAR** When a guitar string is plucked, it is displaced from a fixed point in the middle of the string and vibrates back and forth, producing a musical tone. The exact tone depends on the frequency, or number of cycles per second, that the string vibrates. To produce an A, the frequency is 440 cycles per second, or 440 hertz (Hz).

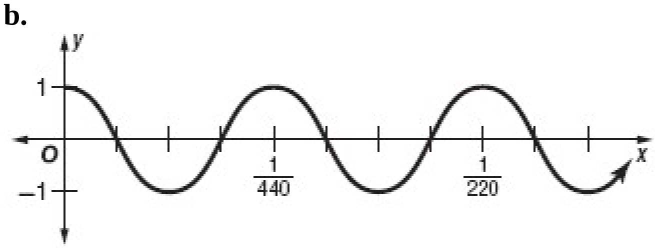
a. Find the period of this function.

b. Graph the height of the fixed point on the string from its resting position as a function of time. Let the maximum distance above the resting position have a value of 1 unit, and let the minimum distance below this position have a value of 1 unit.

SOLUTION:

a. The period of this function is $\frac{1}{440}$ second.

b.



Prove that each statement is true for all positive integers.

53. $4^n - 1$ is divisible by 3.

SOLUTION:

Step 1: $4^1 - 1 = 3$, which is divisible by 3. The statement is true for $n = 1$.

Step 2: Assume that $4^k - 1$ is divisible by 3 for some positive integer k. This means that $4^k - 1 = 3r$ for some whole number r.

Step 3:

$$\begin{aligned} 4^k - 1 &= 3r \\ 4^k &= 3r + 1 \\ 4^{k+1} &= 12r + 4 \\ 4^{k+1} - 1 &= 12r + 3 \\ 4^{k+1} - 1 &= 3(4r + 1) \end{aligned}$$

Since r is a whole number, $4r + 1$ is a whole number. Thus, $4^{k+1} - 1$ is divisible by 3, so the statement is true for $n = k + 1$. Therefore, $4^n - 1$ is divisible by 3 for all positive integers n.

54. $5^n + 3$ is divisible by 4.

SOLUTION:

Step 1: $5^1 + 3 = 8$, which is divisible by 4. The statement is true for $n = 1$.

Step 2: Assume that $5^k + 3$ is divisible by 4 for some positive integer k. This means that $5^k + 3 = 4r$ for some whole number r.

Step 3

$$\begin{aligned} 5^k + 3 &= 4r \\ 5^k &= 4r - 3 \\ 5^{k+1} &= 20r - 15 \\ 5^{k+1} + 3 &= 20r - 12 \\ 5^{k+1} + 3 &= 4(5r - 3) \end{aligned}$$

Since r is a positive integer, $5r - 3$ is a positive integer. Thus, $5^{k+1} + 3$ is divisible by 4, so the statement is true for $n = k + 1$. Therefore, $5^n + 3$ is divisible by 4 for all positive integers n.

Solve each equation.

55. $7 + \sqrt{4x + 8} = 9$

SOLUTION:

$$\begin{aligned} 7 + \sqrt{4x+8} &= 9 \\ \sqrt{4x+8} &= 9 - 7 \\ \sqrt{4x+8} &= 2 \\ \left(\sqrt{4x+8}\right)^2 &= 2^2 \\ 4x + 8 &= 4 \\ 4x &= -4 \\ x &= -1 \end{aligned}$$

56. $\sqrt{y + 21} - 1 = \sqrt{y + 12}$

SOLUTION:

$$\begin{aligned} \sqrt{y+21} - 1 &= \sqrt{y+12} \\ \left(\sqrt{y+21} - 1\right)^2 &= \left(\sqrt{y+12}\right)^2 \\ y + 21 + 1 - 2\sqrt{y+21} &= y + 12 \\ \sqrt{y+21} &= 5 \\ \left(\sqrt{y+21}\right)^2 &= 5^2 \\ y + 21 &= 25 \\ y &= 4 \end{aligned}$$

57. $\sqrt{4z+1} = 3 + \sqrt{4z-2}$

SOLUTION:

$$\begin{aligned}
\sqrt{4z+1} &= 3 + \sqrt{4z-2} \\
\left(\sqrt{4z+1}\right)^2 &= \left(3 + \sqrt{4z-2}\right)^2 \\
4z+1 &= 9 + 4z - 2 + 6\sqrt{4z-2} \\
-6 &= 6\sqrt{4z-2} \\
4z-2 &= 1 \\
4z &= 3 \\
z &= \frac{3}{4}
\end{aligned}$$

Check the solution.

$$\begin{aligned}
\sqrt{4\left(\frac{3}{4}\right)+1} &\stackrel{?}{=} 3 + \sqrt{4\left(\frac{3}{4}\right)-2} \\
\sqrt{3+1} &\stackrel{?}{=} 3 + \sqrt{3-2} \\
2 &= 3+1 \\
2 &\neq 4 \times
\end{aligned}$$

Therefore, there is no solution.